Grassmannians and Equivariant Cohomology

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Introduction

The Grassmannian Manifold $Gr_k(\mathbb{R}^n)$ parametrizes the k dimensional planes through the origin in n-dimensional real space. Identifying the Schubert cell construction of these manifolds allows us to perform a variety of other calculations. Also, we consider group actions, which allow us to investigate the equivariant cohomology and intersections of such grassmannians with C_2 group actions. Others such as [1] have contributed to the theory of Equivariant Cohomology on general manifolds, so we seek to find the necessary extensions to the theory for application to Grassmannian manifolds.

Theory

The Grassmannian Manifold $Gr_k(\mathbb{R}^n)$ is $k(n-k)$-dimensional [2]. A basis for a k-plane in $\mathbb{R}^n$ is a collection of k row vectors of n columns each, forming a matrix. For example, to identify a 1-plane through the origin of $\mathbb{R}^2$, we can use a two column row vector. To identify a 2-plane in $\mathbb{R}^4$, we can use a two row, four column matrix. When writing a k-plane as the rowspace of a matrix, we can always choose a matrix row echelon form. Each family of rowspace matrices with a unique degree of freedom or unique arrangement of its degrees of freedom is identified with a unique Schubert cell[1], notated as Ω with a subscript of the unique Schubert symbol, the sequence of degrees of freedom in the matrix, in accordance with notation from [3]. An equivalent notation to the Schubert symbol is the Young diagram, in which each block

[1]Importantly, we use Schubert cell here really to mean the interior of the cell in consideration, not it's closure

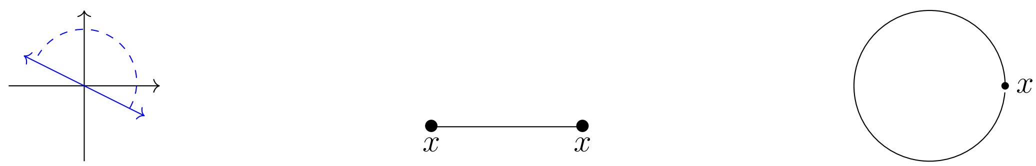


Figure 1: $Gr_1(\mathbb{R}^2)$ visualized three different ways.

represents a degree of freedom in the Schubert cell and each number in the Schubert symbol, left to right, tells us the number of blocks in each row of the Young diagram, top to bottom. For example, a Schubert symbol of $(1,1,2)$ could also be represented as the Young diagram $\ydiagram{1,1,2}$. A Schubert symbol of $(2,3)$ could also be represented as a Young diagram of $\ydiagram{2,3}$. The process of finding the cells (in our case, Schubert cells) which make up a larger manifold is known as cell decomposition.

Example. Consider the construction of $Gr_1(\mathbb{R}^2)$.

As the leftmost item of 1 portrays, $Gr_1(\mathbb{R}^2)$ is the parametrization of 1-d planes (i.e.: lines) through $\mathbb{R}^2$. We can represent these planes as the set of row vectors

$$\{\text{rowspace}[a \;\; b] : a, 0 \neq b \in \mathbb{R}\} = \left\{\text{rowspace}\left[\frac{a}{b} \;\; 1\right] : a, 0 \neq b \in \mathbb{R}\right\} = \Omega_{\square},$$

$$\{\text{rowspace}[a \;\; 0] : 0 \neq a \in \mathbb{R}\} = \left\{\text{rowspace}\left[1 \;\; 0\right]\right\} = \Omega_{*}$$

Because $k = 1$ and $n - k = 1$, we know $\Omega_{\square}$ is the largest Schubert cell in $Gr_1(\mathbb{R}^2)$, which we can also write as $\Omega_{(1)}$. We also observe that the remaining Schubert symbols comprise a filtration of the maximal Schubert symbol [2]. To determine how the cells attach to one another, we take the limit of the rowspace of a matrix of a higher dimensional cell as chosen sets of free variables go to infinity. Thus, for $Gr_1(\mathbb{R}^2)$, we have a Schubert cell $\Omega_{(1)}$ associated with a line, attached at positive and negative infinity to a point, $\Omega_{(0)}$.

Example. For $Gr_1(\mathbb{R}^3)$, we imagine adding a dimension to the graph on the left of 1, in which case our parametrization sweeps out the interior of a half sphere rather than the interior of a half circle. Now we have a maximal Schubert symbol of (2). By filtration property of cell decomposition, we know this must attach to Schubert cells with symbols of (1) and (0). To calculate these attachments, we take the following limits:

$$\lim_{x\to\infty} \text{rowspace} \begin{bmatrix} ax & bx & 1 \end{bmatrix} = \lim_{x\to\infty} \text{rowspace} \begin{bmatrix} \frac{a}{b} & 1 & \frac{1}{bx} \end{bmatrix} = \text{rowspace} \begin{bmatrix} \frac{a}{b} & 1 & 0 \end{bmatrix}$$

$$\lim_{x\to\infty} \text{rowspace} \begin{bmatrix} ax & b & 1 \end{bmatrix} = \lim_{x\to\infty} \text{rowspace} \begin{bmatrix} 1 & \frac{b}{ax} & \frac{1}{ax} \end{bmatrix} = \text{rowspace} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Because we can choose a and b to be positive or negative independently, there will two different pairs of values that yield the same ratio between a and b. This tells us that $\Omega_{(2)}$ attaches to $\Omega_{(1)}$ in a "times 2" map. Our second calculation tells us that $\Omega_{(0)}$ is a point on the boundary of $\Omega_{(2)}$. In fact, this will look as if we attached the interior of a 2 dimensional disk to the rightmost object in 1, wrapping the boundary of the disk around the circle twice. According to [2] $Gr_1(\mathbb{R}^n) \cong \mathbb{R}P^{n-1}$, projective space of dimension $n-1$, so as we increase n from 3 we know that the attachment maps will be "times 2" maps from the boundary of the n dimensional Schubert cell to the space $Gr_1(\mathbb{R}^n - 1)$.

Furthermore, because k and $n-k$ are complementary dimensions in $\mathbb{R}^n, Gr_k(\mathbb{R}^n) \cong Gr_{n-k}(\mathbb{R}^n)$[2], and their submanifolds of equal dimensions will appear as matrix transposes of one another. Before progressing to equivariant cohomology, let us consider another cell decomposition example with a grassmannian of higher dimension.

Example. See 1.

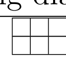
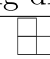
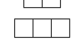
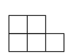
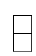
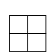
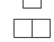
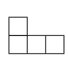
Dimension	Cell	Young diagram	Dimension	Cell	Young diagram
6	$\Omega_{(3,3)}$		3	$\Omega_{(1,2)}$	
				$\Omega_{(0,3)}$	
5	$\Omega_{(2,3)}$				
			2	$\Omega_{(1,1)}$	
4	$\Omega_{(2,2)}$			$\Omega_{(0,2)}$	
			1	$\Omega_{(1)}$	
	$\Omega_{(1,3)}$		0	Ω_*	

Table 1: Cell Decomposition for $Gr_2(\mathbb{R}^5)$

[2] I'm still struggling with writing the homeomorphism here. My brain first went to Poincare duality, and I feel drawn to include that somewhere, but I'm pretty sure this isn't the place.

Once we understand how to calculate cell decomposition in the classical setting, we can consider equivariant topology, specifically formed by applying C_2 group actions to general manifolds. We can apply a $\mathbb{Z}_2$ group action to a grassmannian, cell by cell, in matrix representation by choosing an ordering of n plus and minus signs. This action will be well defined and linear on the Grassmannian because it is a linear action on $\mathbb{R}^n$. For example, the $-+-+$ action on $\mathbb{R}^4$ takes $(w, x, y, z) \mapsto (-w, x, -y, x)$, so the action is linear. Because $-+-+$ is linear on $\mathbb{R}^4, Gr_k(\mathbb{R}^{-+-+})$ inherits a well-defined action. We can simplify the ordering of plus and minus signs to the notation $Gr_k(\mathbb{R}^{p,q})$ where $p = n$ and q is the number of minus signs in the ordering, but this simplification removes the specificity of the ordering (i.e.: $\mathbb{R}^{4,2}$ could correspond to an ordering of $--++$ or $-+-+$). We find the set of fixed points in a given Schubert cell by applying the chosen sign to all values in the corresponding column and simplifying to row echelon form. All degrees of freedom which observe a sign change are unfixed while all degrees of freedom which retain the same sign are fixed. Similar to how we simplify the (p, q) notation when applying group actions to a manifold, we can simplify our notation of the set of fixed points in a given Schubert cell with a bidegree tuple, given by (dimension of cell, number of dimensions within manifold not fixed).[3]

Example: Again, consider the Grassmannian $Gr_2(\mathbb{R}^5)$ with the $-+-+-$ action on $\mathbb{R}^5$. We show the calculation of the fixed points for the cell $\Omega_{\yng(1,3)}$. In the matrix below, a, b, c, d represent arbitrary constants.

$$\begin{matrix} & - & + & - & + & - \end{matrix}$$

$$\text{rowspace}\begin{bmatrix} a & 1 & 0 & 0 & 0 \\ b & 0 & c & d & 1 \end{bmatrix} \mapsto \text{rowspace}\begin{bmatrix} -a & 1 & 0 & 0 & 0 \\ -b & 0 & -c & d & -1 \end{bmatrix}$$

$$= \text{rowspace}\begin{bmatrix} -a & 1 & 0 & 0 & 0 \\ b & 0 & c & -d & -1 \end{bmatrix} = \Omega_{\yng(1,3)}(-a, b, c, -d)$$

which only is fixed if $a = d = 0$, so the set of fixed points in this particular cell is given by $\Omega_{\yng(1,3)}(0, b, c, 0)$.

[3] I think I'm beginning to get what you were saying about needing to do a change of basis for some calculations, but I'm also not nearly confident enough in my understanding o why or how to do it correctly that I'm gonna change anything last minute.

Dimension	Set of Fixed Points	Dimension	Set of Fixed Points
6	$\Omega_{(3,3)}(0,b,0,d,0,f)$	3	$\Omega_{(1,2)}(0,0,0)$
			$\Omega_{(3)}(0,b,0)$
5	$\Omega_{(2,3)}(a,0,c,0,0)$		
		2	$\Omega_{(1,1)}(0,b)$
4	$\Omega_{(2,2)}(a,0,0,d)$		$\Omega_{(2)}(0,b)$
		1	$\Omega_{(1)}(0)$
	$\Omega_{(1,3)}(0,b,c,0)$	0	Ω_{*}

Table 2: The set of fixed points in each submanifold of $Gr_2(\mathbb{R}^{-+-+-})$. Letters fill in alphabetically left to right, top to bottom.

The fixed points become important when calculating the equivariant cohomology of a manifold, a technique which allows us to calculate the intersections of submanifolds with abelian group operations rather than through completely geometric means. According to [1], we must calculate the neighborhoods of the fixed points in each submanifold to find the bidegree of each copy of $\mathbb{M}_2$ in the cohomology. Once we know the bidegrees of the neighborhoods of the fixed points, we can represent the equivariant cohomology of the whole space with a q vs. p graph of upper and lower cones. Depending on the bidegree of the neighborhood of the submanifold, we will have shifted copies of $\mathbb{M}_2$ where the generator occurs at the point on the graph indicated by the bidegree. By generally accepted notation, we shift the origin points on the graph up and to the right.

Example: Let us consider $Gr_1(\mathbb{R}^{3,1})$, visualized by 2 with red colored dots and lines marking the fixed set. As a sample neighborhood calculation, we consider $\Omega_{(1)}$ with the ordering $+-+$ applied. From one of our previous examples, we know that the attachment from the maximum dimensional cell to $\Omega_{(1)}$ is given by

$$\lim_{x\to\infty} \text{rowspace}\begin{bmatrix} ax & bx & 1 \end{bmatrix} = \text{rowspace}\begin{bmatrix} \frac{a}{b} & 1 & 0 \end{bmatrix}$$

Theorem 1.12 in [1] instructs us to inspect the neighborhood of fixed points. In this case, the only fixed point in $\Omega_{(1)}$ is given by rowspace $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, so we know that b must go to infinity "faster" than a. To determine the ratio between how fast each goes to infinity, we look to the set of fixed points in

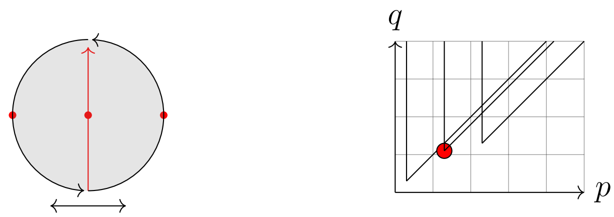


Figure 2: $Gr_1(\mathbb{R}^3)$ with a reflection action and its cohomology graph.

the maximal cell, $\Omega_{(2)}$ with the same ordering applied, which in this case is rowspace $\begin{bmatrix} a & 0 & 1 \end{bmatrix}$. In other words, we need a to stay fixed when we apply the action to the neighborhood of $\Omega_{(1)}$ while b changes signs, so our neighborhood of $\Omega_{(1)}$ is given by rowspace $\begin{bmatrix} ar^2 & br^3 & 1 \end{bmatrix}$. Thus, $N(\Omega_{(1)}) \cong \Omega_{(1)} \times \mathbb{R}^{1,1}$ so the the cohomology of $Gr_1(\mathbb{R}^{3,1})$ has an $\mathbb{M}_2$ generator at the coordinates $(1,1)$ on our graph, highlighted with a red dot on the graph in 2, which displays the entire cohomology of $Gr_1(\mathbb{R}^3)$.

Calculations

$Gr_2(\mathbb{R}^{4,1})$ Calculations (r can be treated as a dimension of freedom, while any x or y is a fixed number.):

$\mathrm{Gr}_2(\mathbb{R}^{-+++})$	Fixed set	Neighborhood of fixed points	Tangent bundle
$\ytableaushort{ab,cd}$	$\Omega_{\ydiagram{2,2}}(0,b,0,d)$	$\emptyset$	$\mathbb{R}^{0,0}$
$\ytableaushort{a,bc}$	$\Omega_{\ydiagram{1,2}}(0,0,c)$	$\Omega^{1,0}_{\ydiagram{2,2}}(a,r,-xa,-rx)$	$\mathbb{R}^{2,1}$
$\ytableaushort{a,b}$	$\Omega_{\ydiagram{1,1}}(0,0)$	$\Omega^{1,1}_{\ydiagram{2,2}}(0,b,r,r^2)$	$\mathbb{R}^{2,1}$
$\ytableaushort{ab}$	$\Omega_{\ydiagram{2}}(a,b)$	$\Omega^{1,0}_{\ydiagram{2,2}}(a,b,ax,bx)$	$\mathbb{R}^{2,1}$
$\ytableaushort{a}$	$\Omega_{\ydiagram{1}}(a)$	$\Omega^{1,1}_{\ydiagram{2,2}}(a,b,c,d)$	$\mathbb{R}^{3,2}$
pt		$\Omega^{1,1}_{\ydiagram{2,2}}(r,r^2,r,r^2)$	$\mathbb{R}^{4,2}$

$\mathrm{Gr}_2(\mathbb{R}^{+-++})$	Fixed set	Neighborhood of fixed points	Tangent bundle
$\ytableaushort{ab,cd}$	$\Omega_{\ydiagram{2,2}}(a,0,c,0)$	$\emptyset$	$\mathbb{R}^{0,0}$
$\ytableaushort{a,bc}$	$\Omega_{\ydiagram{1,2}}(0,b,c)$	$\Omega^{1,0}_{\ydiagram{2,2}}(a,r,-ax+y,-rx)$	$\mathbb{R}^{2,1}$
$\ytableaushort{a,b}$	$\Omega_{\ydiagram{1,1}}(0,b)$	$\Omega^{1,1}_{\ydiagram{2,2}}(0,b,r,r^2)$	$\mathbb{R}^{2,2}$
$\ytableaushort{ab}$	$\Omega_{\ydiagram{2}}(0,b)$	$\Omega^{1,0}_{\ydiagram{2,2}}(r,-rx,b,-bx)$	$\mathbb{R}^{2,1}$
$\ytableaushort{a}$	$\Omega_{\ydiagram{1}}(0)$	$\Omega^{1,1}_{\ydiagram{2,2}}(a,0,r,d)$	$\mathbb{R}^{3,1}$
pt		$\Omega^{1,1}_{\ydiagram{2,2}}(r^2,r,r^2,r)$	$\mathbb{R}^{4,2}$

$\mathrm{Gr}_2(\mathbb{R}^{++-+})$	Fixed set	Neighborhood of fixed points	Tangent bundle
$\ytableaushort{ab,cd}$	$\Omega_{\ydiagram{2,2}}(0,0,c,d)$	$\emptyset$	$\mathbb{R}^{0,0}$
$\ytableaushort{a,bc}$	$\Omega_{\ydiagram{1,2}}(a,b,0)$	$\Omega^{1,0}_{\ydiagram{2,2}}(r,rx,0,0)$	$\mathbb{R}^{2,1}$
$\ytableaushort{a,b}$	$\Omega_{\ydiagram{1,1}}(a,0)$	$\Omega^{1,1}_{\ydiagram{2,2}}(bx,b,rx,r)$	$\mathbb{R}^{2,1}$
$\ytableaushort{ab}$	$\Omega_{\ydiagram{2}}(a,0)$	$\Omega^{1,0}_{\ydiagram{2,2}}(r,b,0,x)$	$\mathbb{R}^{2,2}$
$\ytableaushort{a}$	$\Omega_{\ydiagram{1}}(0)$	$\Omega^{1,1}_{\ydiagram{2,2}}(a,0,r,d)$	$\mathbb{R}^{3,1}$
pt		$\Omega^{1,1}_{\ydiagram{2,2}}(0,0,c,d)$	$\mathbb{R}^{4,2}$

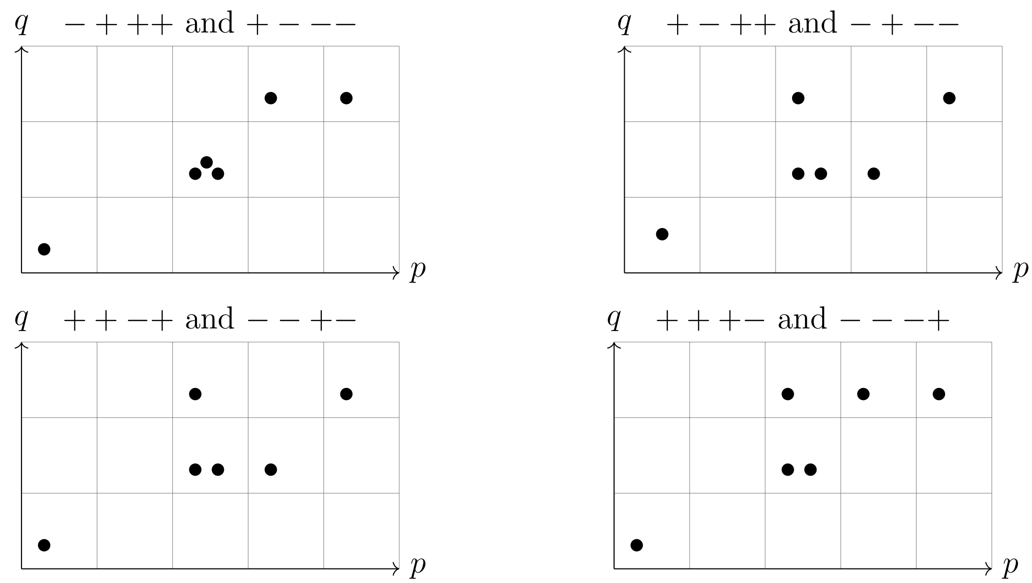


Figure 3: Graphical representation of the calculations for $Gr_2(\mathbb{R}^{4,1}) \cong Gr_2(\mathbb{R}^{4,3}$.

$\mathrm{Gr}_2(\mathbb{R}^{+++-})$	Fixed set	Neighborhood of fixed points	Tangent bundle
$\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}$	$\Omega_{\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}}(a,b,0,0)$	$\emptyset$	$\mathbb{R}^{0,0}$
$\begin{smallmatrix} a & \\ b & c \end{smallmatrix}$	$\Omega_{\begin{smallmatrix}\square&\\\square&\square\end{smallmatrix}}(a,0,0)$	$\Omega^{1,0}_{\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}}(r,rx,d,dx)$	$\mathbb{R}^{2,1}$
$\begin{smallmatrix} a \\ b \end{smallmatrix}$	$\Omega_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(a,b)$	$\Omega^{1,1}_{\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}}(ar^2,r^2,br-ar^3,-r^3)$	$\mathbb{R}^{2,2}$
$\begin{smallmatrix} a & b \end{smallmatrix}$	$\Omega_{\begin{smallmatrix}\square&\square\end{smallmatrix}}(0,0)$	$\Omega^{1,0}_{\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}}(r^2,b,-r,0)$	$\mathbb{R}^{2,1}$
$\begin{smallmatrix} a \end{smallmatrix}$	$\Omega_{\square}(a)$	$\Omega^{1,1}_{\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}}(a,b,r,d)$	$\mathbb{R}^{3,2}$
pt		$\Omega^{1,1}_{\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}}(a,b,c,d)$	$\mathbb{R}^{4,2}$

$Gr_2(\mathbb{R}^{4,2})$ calculations:

$\mathrm{Gr}_2(\mathbb{R}^{--++})$	Fixed set	Neighborhood of fixed points	Tangent bundle
$\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}$	$\Omega_{\begin{smallmatrix}\square&\square\\ \square&\square\end{smallmatrix}}(0,0,0,0)$	$\emptyset$	$\mathbb{R}^{0,0}$
$\begin{smallmatrix} a & \\ b & c \end{smallmatrix}$	$\Omega_{\begin{smallmatrix}\square&\\ \square&\square\end{smallmatrix}}(a,0,c)$	$\Omega^{1,0}_{\begin{smallmatrix}\square&\square\\ \square&\square\end{smallmatrix}}(a,r,-ax,-rx)$ or (rx,r,dx,d)	$\mathbb{R}^{2,2}$
$\begin{smallmatrix} a \\ b \end{smallmatrix}$	$\Omega_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}(a,0)$	$\Omega^{1,1}_{\begin{smallmatrix}\square&\square\\ \square&\square\end{smallmatrix}}(bx,b,rx,r)$	$\mathbb{R}^{2,2}$
$\begin{smallmatrix} a & b \end{smallmatrix}$	$\Omega_{\square\square}(0,b)$	$\Omega^{1,0}_{\begin{smallmatrix}\square&\square\\ \square&\square\end{smallmatrix}}(r,b,rx,bx)$	$\mathbb{R}^{2,2}$
a	$\Omega_{\square}(0)$	$\Omega^{1,1}_{\begin{smallmatrix}\square&\square\\ \square&\square\end{smallmatrix}}(a,0,r,d)$	$\mathbb{R}^{3,3}$
pt		$\Omega^{1,1}_{\begin{smallmatrix}\square&\square\\ \square&\square\end{smallmatrix}}(a,b,c,d)$	$\mathbb{R}^{4,4}$

$\mathrm{Gr}_2(\mathbb{R}^{-+-+})$	Fixed set	Neighborhood of fixed points	Tangent bundle
$\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}$	$\Omega_{\begin{smallmatrix}\square&\square\\ \square&\square\end{smallmatrix}}(a,0,0,d)$	$\emptyset$	$\mathbb{R}^{0,0}$
$\begin{smallmatrix} a & \\ b & c \end{smallmatrix}$	$\Omega_{\begin{smallmatrix}\square&\\ \square&\square\end{smallmatrix}}(0,0,0)$	$\Omega^{1,0}_{\begin{smallmatrix}\square&\square\\ \square&\square\end{smallmatrix}}(a,r,0,0)$ or $(0,r,0,d)$	$\mathbb{R}^{2,1}$
$\begin{smallmatrix} a \\ b \end{smallmatrix}$	$\Omega_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}(0,b)$	$\Omega^{1,1}_{\begin{smallmatrix}\square&\square\\ \square&\square\end{smallmatrix}}(x,b,r,0)$	$\mathbb{R}^{2,2}$
$\begin{smallmatrix} a & b \end{smallmatrix}$	$\Omega_{\square\square}(a,0)$	$\Omega^{1,0}_{\begin{smallmatrix}\square&\square\\ \square&\square\end{smallmatrix}}(r^2,b,-r,x)$	$\mathbb{R}^{2,2}$
a	$\Omega_{\square}(0)$	$\Omega^{1,1}_{\begin{smallmatrix}\square&\square\\ \square&\square\end{smallmatrix}}(a,0,r,d)$	$\mathbb{R}^{3,1}$
pt		$\Omega^{1,1}_{\begin{smallmatrix}\square&\square\\ \square&\square\end{smallmatrix}}(a,b,c,d)$	$\mathbb{R}^{4,2}$

$\mathrm{Gr}_2(\mathbb{R}^{-++-})$	Fixed set	Neighborhood of fixed points	Tangent bundle
$\begin{ytableau} a & b \\ c & d \end{ytableau}$	$\Omega_{\ydiagram{2,2}}(0,b,c,0)$	$\emptyset$	$\mathbb{R}^{0,0}$
$\begin{ytableau} a \\ b & c \end{ytableau}$	$\Omega_{\ydiagram{1,2}}(0,b,0)$	$\Omega^{1,0}_{\ydiagram{2,2}}(r,r^2,x,d)$ or (a,r^2,x,r)	$\mathbb{R}^{2,2}$
$\begin{ytableau} a \\ b \end{ytableau}$	$\Omega_{\ydiagram{1,1}}(0,0)$	$\Omega^{1,1}_{\ydiagram{2,2}}(0,b,0,r)$	$\mathbb{R}^{2,1}$
$\begin{ytableau} a & b \end{ytableau}$	$\Omega_{\ydiagram{2}}(0,0)$	$\Omega^{1,0}_{\ydiagram{2,2}}(r,b,0,0)$	$\mathbb{R}^{2,1}$
$\begin{ytableau} a \end{ytableau}$	$\Omega_{\ydiagram{1}}(a)$	$\Omega^{1,1}_{\ydiagram{2,2}}(a,b,r,d)$	$\mathbb{R}^{3,2}$
pt		$\Omega^{1,1}_{\ydiagram{2,2}}(a,b,c,d)$	$\mathbb{R}^{4,2}$

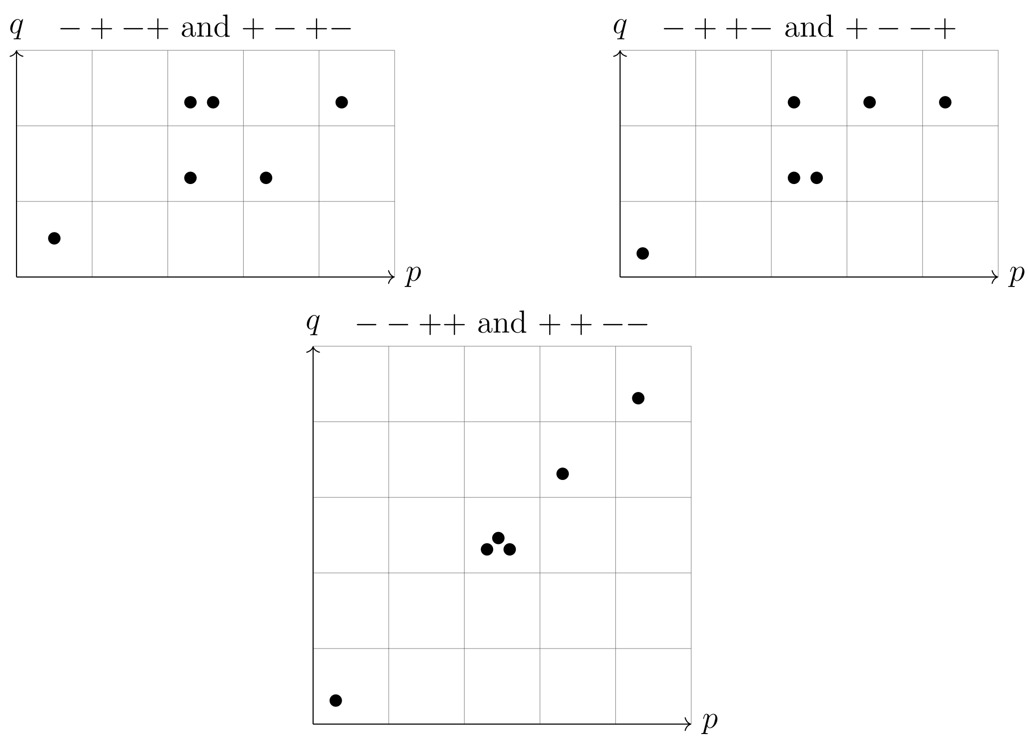


Figure 4: Graphical representation of the calculations for $Gr_2(\mathbb{R}^{4,2})$.

Conclusion

Because our calculations for each action do not agree, shown in the discrepancy in the graphs in 3 and 4, further research is required to determine how to appropriately calculate the cohomology of $Gr_2(\mathbb{R}^{4,2})$. Our results also appear misleading because they suggest that the "L-block" cell (the only cell with three degrees of freedom) has a 2-dimensional neighborhood, when the dimension of the entire Grassmannian only makes it possible for a cell with three degrees of freedom to have a 1-dimensional neighborhood. Furthermore, more research is required to discover a more generic way to attach the L-block to our 4-dimensional Schubert cell, as in $Gr_2(\mathbb{R}^{4,2})$ we had two limits that didn't attach completely generally, but each yielded results.

Also, as the exact bidegree of the neighborhood of each submanifold changes with the ordering of plus and minus signs, determining the submanifolds that must be distinguished when performing equivariant intersections requires either more thorough examination of the neighborhoods or a better geometric visualization of the space, in order to apply concepts from [1].

References

[1] Hazel, Christy (2019) *Equivariant Fundamental Classes in $RO(C_2)$-Graded Cohomology in $\mathbb{Z}/2$ Coefficients.*

[2] Hopkins, M.J. (December 5, 2010) *Grassmannian Manifolds.*

[3] Manivel, Laurent. (1998) *Symmetric Functions, Schubert Polynomials, and Degeneracy Loci.*