Equivariant Intersection Theory on Grassmanian Manifolds

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Introduction

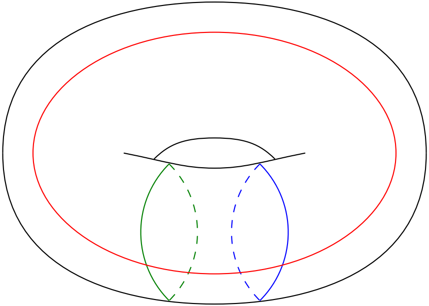
- Topological Spaces
- Intersecting Different Spaces
- Space with (Group) Actions
- Grassmannians

Examples

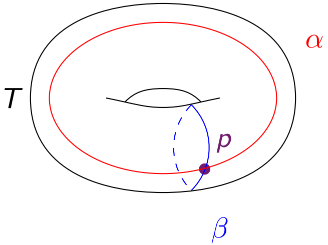
We explore Topological spaces (Grassmannians) and how they intersect (up to homotopy) and the behavior of their intersection when we apply group actions ($\mathbb{Z}_2$ actions).

Intersection Theory

To understand classical intersection theory, we'll start with the submanifolds of a Torus. First we'll explain briefly the concept of homotopy.



Intersection Theory

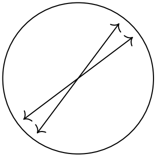


$\cap$	T	α	β	p
T	T	α	β	p
α	α	$\emptyset$	p	$\emptyset$
β	β	p	$\emptyset$	$\emptyset$
p	p	$\emptyset$	$\emptyset$	$\emptyset$

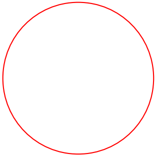
Table 1: Minimal intersection of submanifolds of a Torus.

Group Actions

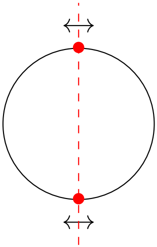
Applying $\mathbb{Z}_2$ group actions can reflect, spin, or otherwise move a topological space. It also requires us to keep track of the set of fixed points(the fixed set).



$S^1_{\text{antipodal}}$



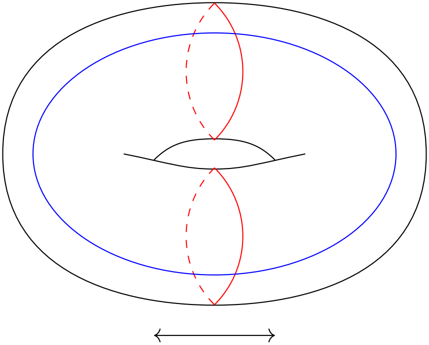
S^1_{trivial}



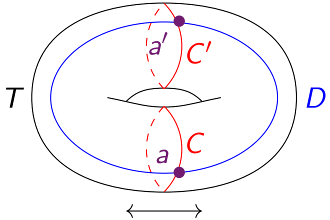
S^1_{flip}

Equivariant Intersection Theory

We can also intersect the subspaces of spaces with group actions (known as equivariant intersection theory).



Equivariant Intersection Theory



$\cap$	T	C	C'	D	a	a'
T	T	C	C'	D	a	a'
C	C	C	$\emptyset$	a	$\emptyset$	$\emptyset$
C'	C'	$\emptyset$	C'	a'	$\emptyset$	$\emptyset$
D	D	a	a'	$\emptyset$	$\emptyset$	$\emptyset$
a	a	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
a'	a'	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Table 2: Minimal Equivariant Intersection of Submanifolds of a Torus.

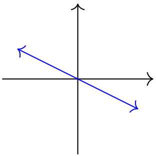
Grassmanians

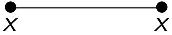
The topological space we're interested in studying equivariantly is called a Grassmannian manifold.

Definition

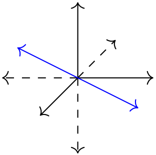
The Grassmannian $Gr_k\mathbb{R}^n$ is the set of all k-dimensional linear subspaces of a n-dimensional vector space.

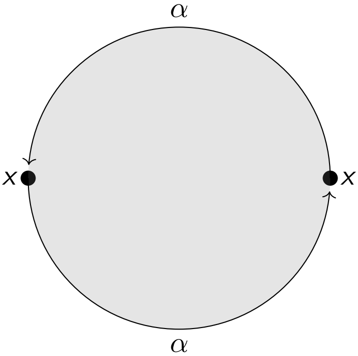
Grassmannians





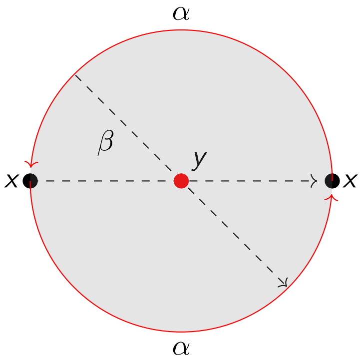
$\mathrm{Gr}_1\mathbb{R}^2$



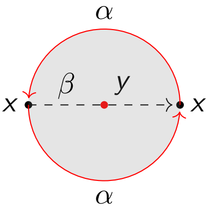


$\mathrm{Gr}_1\mathbb{R}^3$

Equivariant Action on a Grassmannian



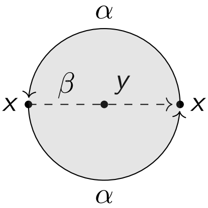
Equivariant Intersection on a Grassmannian



$\cap$	$Gr_1(\mathbb{R}^3)$	α	β	x	y
$Gr_1(\mathbb{R}^3)$	$Gr_1(\mathbb{R}^3)$	α	β	x	y
α	α	α	x	x	$\emptyset$
β	β	x	y	$\emptyset$	y
x	x	x	$\emptyset$	$\emptyset$	$\emptyset$
y	y	$\emptyset$	y	$\emptyset$	y

Table 3: Minimal Equivariant Intersection of Submanifolds of $Gr_1(\mathbb{R}^3)$.Here $\alpha \neq \beta, x \neq y$.

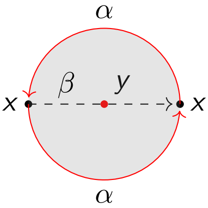
Classical Intersection on a Grassmannian



$\cap$	$Gr_1(\mathbb{R}^3)$	α	β	x	y
$Gr_1(\mathbb{R}^3)$	$Gr_1(\mathbb{R}^3)$	α	β	x	y
α	α	x	x	$\emptyset$	$\emptyset$
β	β	x	x	$\emptyset$	$\emptyset$
x	x	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
y	y	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Table 4: Minimal Classical Intersection of Submanifolds of $Gr_1(\mathbb{R}^3)$. Here $\alpha = \beta, x = y$.

Equivariant Intersection on a Grassmannian



$\cup$	$[Gr_1(\mathbb{R}^3)]$	$[\alpha]$	$[\beta]$	$[x]$	$[y]$
$[Gr_1(\mathbb{R}^3)]$	$[Gr_1(\mathbb{R}^3)]$	$[\alpha]$	$[\beta]$	$[x]$	$[y]$
$[\alpha]$	$[\alpha]$	$\rho[\alpha] + \tau[x]$	$\tau[x]$	$\rho[x]$	0
$[\beta]$	$[\beta]$	$\tau[x]$	$\tau[y]$	0	$\rho[y]$
$[x]$	$[x]$	$\rho[x]$	0	0	0
$[y]$	$[y]$	0	$\rho[y]$	0	$\rho^2[y]$

Table 5: Combined Table for Equivariant Intersection of Submanifolds of $Gr_1(\mathbb{R}^3)$. $[\alpha] = [\beta] + \rho$. Setting $\rho = 0$ and $\tau = 1$ recovers the non-equivariant table.

Future Work/Our Work

- Using classical intersection to solve for the relationships of submanifolds of Grassmannians in equivariant intersection
- Specify equivariant cohomology and intersection theory for Grassmannians
- Specify the attachment maps for Grassmannian submanifolds

Acknowledgements

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